Manifolds of Positive Scalar Curvature and Conformal Cobordism Theory

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February 1, 2008

Abstract

We study here compact manifolds with positive scalar curvature metrics. We use the relative Yamabe invariant from [1] to define the conformal cobordism relation on the category of such manifolds. We prove that corresponding conformal cobordism groups $\mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$ are isomorphic to the cobordism groups $\mathcal{P}\mathrm{os}_n(\gamma)$ defined topologically by S. Stolz in [16]. As a corollary we show that the conformal concordance of positive scalar curvature metrics coincides with the standard concordance relation. Our main technical tools came from the analysis and conformal geometry.

1 Introduction

1.1. Motivation. There are two competing approaches in the study of manifolds admitting a metric of positive scalar curvature.

The first approach is developed within conformal geometry and analysis, and the second one unconventionally may be called "topological" (where the *Spin*-geometry and the Dirac operator methods are combined with the differential topology and some homotopy theory). There are recent detailed surveys presenting a current state of affairs in the subject, given by M. Gromov [8], and J. Rosenberg & S. Stolz [14]. It is emphasized in [8], that the conformal geometry technique (which, perhaps, includes the minimal surface method) has certain advantages over the topological methods since it does not require *Spin* structure and, in some respect, even completeness of a manifold. "On the other hand, whenever the Dirac method applies it delivers finer geometrical (and topological) information although in no serious case the results of one method may be completely recaptured by the other."**

The goal of this paper is to establish one particular link between the topological and conformal approaches, where the resulting object is, indeed, the same. Namely, we show that the cobordism groups of manifolds with positive scalar curvature metrics, delivered by topological means and by means of conformal geometry, coincide.

*Partialy supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan, No. 09640102
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**We quote [8, p. 45]

1.2. Restictions. We restrict here our attention to the oriented smooth manifolds. There are also the dimensional restrictions: all topological constructions work well starting with dimension five for closed manifolds (and six for manifolds with boundary). The conformal geometry gives the dimensional restiction at least two (for closed manifolds) and at least three otherwise. We use abbreviation "psc" for positive scalar curvature.

1.3. Topological psc-cobordism. Let (M_0, g_0), (M_1, g_1) be compact manifolds with psc-metrics g_0 and g_1. Then (M_0, g_0), (M_1, g_1) are *psc-cobordant* if there exists a Riemannian manifold $(W, \bar{g})$, $\partial W = M_0 \sqcup (-M_1)$, so that

$$R_{\bar{g}} > 0, \quad \bar{g}|_{M_j} = g_j, \quad \text{ and } \bar{g} = g_j + dt^2 \text{ near the boundary } \partial W = M_0 \sqcup M_1 \text{ for } j = 0, 1.$$

We emphasize the importance of the condition that the metric $\bar{g}$ must be a product metric near the boundary. In the case of *Spin* manifolds it gives, in particular, that the Dirac operator with the Atiyah-Patodi-Singer boundary conditions is well-defined. In fact, as it was showed by S. Stolz [16], each given manifold M (admitting a psc-metric, and not necessarily *Spin*) lives in a specific cobordism category $\mathfrak{Pos}(\gamma)$, $\gamma = (\pi, w, \hat{\pi})$, determined by the fundametal group $\pi = \pi_1(M)$ and the Stiefel-Whitney classes $w_1(M)$, $w_2(M)$. In particular, $w : \pi \to \mathbf{Z}_2$ is nothing but the orientation character given by $w_1(M)$. We say that the stucture $\gamma = (\pi, w, \hat{\pi})$ is *oriented* if $w = 0$. Let $\mathcal{P}\mathrm{os}_n(\gamma)$ be the corresponding psc-cobordism groups.

1.4. Conformal cobordism. Let (M, g) be a compact manifold with psc-metric g. In the conformal geometry world, it means that the conformal class $C = [g]$ is such that the Yamabe constant $Y_C(M) > 0$. We call such a conformal class C *positive*. Let $\mathcal{C}^+(M)$ be the space of positive conformal classes. We call a pair (M, C) with $C \in \mathcal{C}^+(M)$ a *positive conformal manifold*. Now let W be a compact smooth manifold with boundary, $\partial W = M \neq \emptyset$, and let C be a conformal class on M. Let $\bar{C}$ be a conformal class on W. We write $\partial \bar{C} = C$ if the conformal class $\bar{C}$ is such that $\bar{C}|_M = C$. We defined in [1] the *relative Yamabe constant* $Y_{\bar{C}}(W, M; C)$ and the *relative Yamabe invariant*

$$Y(W, M; C) = \sup_{\bar{C}, \partial \bar{C} = C} Y_{\bar{C}}(W, M; C).$$

We emphasize that in order to define $Y_{\bar{C}}(W, M; C)$, we use the subclass $\bar{C}^0 \subset \bar{C}$ of metrics with *zero mean curvature* along M (see [1] and Section 2).

Positive conformal manifolds (M_0, C_0), (M_1, C_1) are *conformally cobordant* if there exists a smooth cobordism W with boundary $\partial W = M_0 \sqcup (-M_1)$, and such that the relative Yamabe invariant $Y(W, M_0 \sqcup M_1; C_0 \sqcup C_1) > 0$. We proved in [1] that the conformal cobordism is an equivalence relation. We also incorporate the above oriented γ-structure into this cobordism equivalence to define the conformal cobordism groups $\mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$ of positive conformal manifolds equipped with a γ-structure. Clearly there is a natural homomorphism $\mathcal{P}\mathrm{os}_n(\gamma) \longrightarrow \mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$ given by taking conformal classes of corresponding metrics.

Remark. Perhaps, it is important to emphasize the major difference between the above cobordism relations. Firstly, it is in the boundary conditions: product metric near the

boundary versus vanishing of the mean curvature along the boundary. Secondly, let $\bar{C}$ be a conformal class on W such that the relative Yamabe constant $Y_{\bar{C}}(W, M_0 \sqcup M_1; C_0 \sqcup C_1)$ is positive: such a conformal class $\bar{C}$ exists if the relative Yamabe invariant $Y(W, M_0 \sqcup M_1; C_0 \sqcup C_1) > 0$. Then a metric $\bar{g} \in \bar{C}$ (which restricts to given psc-metrics g_0 and g_1 on the boundary and even is a product metric near the boundary) may not have, in general, positive scalar curvature.

1.5. Main results.

Theorem A. *Let γ be an oriented structure, and $n \geq 5$. Then the conformal cobordism groups $\mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$ are naturally isomorphic to the psc-cobordism groups $\mathcal{P}\mathrm{os}_n(\gamma)$.*

Recall that in the conformal world the classic Yamabe invariant $Y(M)$ gives very simple answer on the existence of psc-metric. Indeed:

- *Let M be a closed oriented manifold with $\dim M \geq 2$. Then the Yamabe invariant $Y(M) > 0$ if and only if there exists a psc-metric on M.*

The relative Yamabe invariant has a similar property (where the manifolds below are oriented).

Corollary B. *Let $M = \partial W$, $\dim M \geq 2$, and g be a psc-metric on M. Then the relative Yamabe invariant $Y(W, M; [g]) > 0$ if and only if the metric g may be extended to a psc-metric $\bar{g}$ on W, so that $\bar{g}$ is a product metric near the boundary.*

On the topological side of this story, S. Stolz also defines the relative cobordism groups $R_n(\gamma)$, [16] (see [9] for the simply connected case). S. Stolz proves that the cobordism groups $R_n(\gamma)$ are the *actual obstruction groups* for the existence of psc-metrics (see [16, Theorem 1.1]). The groups $\mathcal{P}\mathrm{os}_n(\gamma)$, $R_n(\gamma)$ and the regular cobordism groups $\Omega_n(\gamma)$ (of manifolds carrying γ-structure) fit together into the exact sequence

$$\cdots \to R_{n+1}(\gamma) \longrightarrow \mathcal{P}\mathrm{os}_n(\gamma) \longrightarrow \Omega_n(\gamma) \longrightarrow R_n(\gamma) \longrightarrow \mathcal{P}\mathrm{os}_{n-1}(\gamma) \to \cdots \tag{1}$$

In the case of simply connected *Spin* manifolds, $\Omega_n(\gamma) = \Omega_n^{\mathrm{Spin}}$. We define the conformal "relatives" to $R_n(\gamma)$ (the cobordism groups $R_n^{\mathrm{conf}}(\gamma)$) for oriented γ-structures, so that there is the exact sequence

$$\cdots \to R_{n+1}^{\mathrm{conf}}(\gamma) \longrightarrow \mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma) \longrightarrow \Omega_n(\gamma) \longrightarrow R_n^{\mathrm{conf}}(\gamma) \longrightarrow \mathcal{P}\mathrm{os}_{n-1}^{\mathrm{conf}}(\gamma) \to \cdots$$

which turns out to be isomorphic to (1). In particular, we have

Corollary C. *Let γ be an oriented structure, and $n \geq 6$. Then the conformal cobordism groups $R_n^{\mathrm{conf}}(\gamma)$ are naturally isomorphic to the psc-cobordism groups $R_n(\gamma)$.*

1.6. Concordance and conformal concordance of psc-metrics. Recall that two psc-metrics g_0, g_1 on M are *psc-concordant* if there exists a psc-metric $\bar{g}$ on a cylinder $M \times [\ell_0, \ell_1]$ (for some $\ell_0 < \ell_1$) so that

$$\bar{g}|_{M \times \{\ell_j\}} = g_j, \quad \text{and } \bar{g} = g_j + dt^2 \text{ near the boundary } M \times \{\ell_j\} \text{ for } j = 0, 1.$$

Two positive conformal classes $C_0, C_1 \in \mathcal{C}^+(M)$ are *conformally concordant* if the Yamabe invariant

$$Y(M \times [0,1], M \times \{0,1\}\,; C_0 \sqcup C_1) > 0.$$

We proved in [1] that conformal concordance is an equivalence relation. Clearly the psc-concordance implies the conformal concordance.

Corollary D. *Let M be an oriented manifold with $\dim M \geq 2$, and let g_0, g_1 be psc-metrics on M such that the conformal classes $C_0 = [g_0]$ and $C_1 = [g_1]$ are conformally concordant. Then the metrics g_0, g_1 are psc-concordant.*

1.7. Important remark. Unfortunately the results of this paper do not allow to compute the conformal cobordism groups $\mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$ and $R_n^{\mathrm{conf}}(\gamma)$. However, in our view, these results open up some new possiblities which hopefully will be explored by open-minded geometers and topologists. We discuss this in Section 6.

1.8. Organization of the paper. We review necessary constructions and facts on the conformal geometry in Section 2. We state our main technical result in Section 3 and outline key points of its proof. We give this proof in Section 4. We review some topological constructions and finish the proofs in Section 5. Finally we discuss some open problems in Section 6.

1.9. Acknowledgments. Both authors would like to acknowledge partial financial support provided by the Department of Mathematics at the University of Oregon, SFB 478 – Geometrische Strukturen in der Mathematik, and the Grants-in-Aid for Scientific Research, Japan. We would like also to thank Michael Joachim, Wolfgang Lück, and Thomas Schick for useful discussions and warm hospitality during our visit at Münster.

2 Some conformal geometry

2.1. General setting. Let W be a compact smooth manifold with boundary, $\partial W = M \neq \emptyset$, and $n = \dim W \geq 3$. We always assume that all manifolds are oriented, and the orientation on W is compatible with the orientaion on its boundary ∂W.

Let C be a conformal class of metrics on M, and $\mathcal{R}\mathrm{iem}(W)$ is the space of all Riemannian metrics on W. For a metric $\bar{g} \in \mathcal{R}\mathrm{iem}(W)$ we denote $H_{\bar{g}}$ the mean curvature along the boundary $\partial W = M$. We denote $\mathcal{C}(M)$ and $\mathcal{C}(W)$ the space of conformal classes on M and W respectively. Let $C \in \mathcal{C}(M)$ $\bar{C} \in \mathcal{C}(W)$. We say that C is the *boundary of* $\bar{C}$ or $\bar{C}$ is a *coboundary of* C if $\bar{C}|_M = C$. We use notation $\partial\bar{C} = C$ in this case. Then a pair of conformal classes $(\bar{C}, C)$ is a *conformal class on* (W, M) *if* $\partial\bar{C} = C$. We denote $\mathcal{C}(W, M)$ the space of pairs of conformal classes. Let $(\bar{C}, C) \in \mathcal{C}(W, M)$. For each pair of conformal classes $(\bar{C}, C) \in \mathcal{C}(W, M)$ we consider the conformal subclass $\bar{C}^0 \subset \bar{C}$ defined as

$$\bar{C}^0 = \left\{ \bar{g} \in \bar{C} \mid H_{\bar{g}} = 0 \right\}.$$

We call $\bar{C}^0 \subset \bar{C}$ the *normalized conformal class*. Let $\mathcal{C}^0(W, M)$ be the space of pairs $(\bar{C}^0, C)$, so that $\bar{C}^0 \subset \bar{C}$ as above, and $(\bar{C}, C) \in \mathcal{C}(W, M)$. In fact, it is easy to see that for any conformal class $\bar{C} \in \mathcal{C}(W)$ the subclass $\bar{C}^0$ is not empty (see [5, formula (1.4)]).

Thus there is a natural bijection between the spaces $\mathcal{C}^0(W,M)$ and $\mathcal{C}(W,M)$. Let $\bar{g} \in \bar{C}^0$ be a metric. Then $\bar{C}^0$ may be described as follows:

$$\bar{C}^0 = \left\{ u^{\frac{4}{n-2}} \bar{g} \mid u \in C^\infty_+(W) \text{ such that } \ \partial_\nu u = 0 \ \text{ along } M \ \right\}.$$

Here ν is a normal unit (inward) vector field along the boundary, and $C^\infty_+(W)$ is the space of positive smooth functions on W.

2.2. The Einstein-Hilbert functional. Let $C \in \mathcal{C}(M)$ be given. We define the following subspaces of metrics:

$$\begin{aligned} \mathcal{R}\mathrm{iem}_C(W,M) &= \{\bar{g} \in \mathcal{R}\mathrm{iem}(W) \mid \partial[\bar{g}] = C\}, \\ \mathcal{R}\mathrm{iem}^0_C(W,M) &= \{\bar{g} \in \mathcal{R}\mathrm{iem}_C(W) \mid H_{\bar{g}} = 0\}. \end{aligned}$$

The normalized Einstein-Hilbert functional $I : \mathcal{R}\mathrm{iem}^0_C(W,M) \longrightarrow \mathbf{R}$ is given by

$$I(\bar{g}) = \frac{\int_W R_{\bar{g}} d\sigma_{\bar{g}}}{\mathrm{Vol}_{\bar{g}}(W)^{\frac{n-2}{n}}},$$

where $R_{\bar{g}}$ is the scalar curvature, and $d\sigma_{\bar{g}}$ is the volume element. The following fact is analogous to the classic theorem on the Einstein-Hilbert functional.

Theorem 2.1. ([1, Theorem 1.1]) *Critical points of the functional I on the space of metrics $\mathcal{R}\mathrm{iem}^0_C(W,M)$ coincide with the set of Einstein metrics $\bar{g}$ on W with $\partial[\bar{g}] = C$, and $H_{\bar{g}} = 0$.*

2.3. Relative Yamabe invariant. Let $(\bar{C}, C) \in \mathcal{C}(W,M)$. The *relative Yamabe costant of* $(\bar{C}, C)$ is defined as

$$Y_{\bar{C}}(W,M;C) = \inf_{\bar{g} \in \bar{C}^0} I(\bar{g}).$$

Remark. The relative Yamabe constant $Y_{\bar{C}}(W,M;C)$ is related to the Yamabe problem on a manifold with boundary, which was solved by P. Cherrier [4] and J. Escobar [5] under some restrictions. In fact, in a generic case there is a relative *Yamabe metric* $\check{g} \in \bar{C}$ with $H_{\check{g}} = 0$ and constant scalar curvature $R_{\check{g}} = Y_{\bar{C}}(W,M;C) \cdot \mathrm{Vol}_{\check{g}}(W)^{-\frac{2}{n}}$ (see [5], [1] for more details).

The *relative Yamabe invariant* with respect to a conformal class $C \in \mathcal{C}(M)$ is defined as:

$$Y(W,M;C) = \sup_{\bar{C}, \partial\bar{C} = C} Y_{\bar{C}}(W,M;C).$$

The relative Yamabe invariant $Y(W,M;C)$ has several important properties analogous to the corresponding properties of the classic Yamabe invariant (see [1] for details).

2.4. Approximation theorem. One notices that the minimal boundary condition is crucial to define the relative Yamabe constant. In general, it is rather delicate problem to approximate given conformal class $\bar{C}$ on a manifold with boundary by such conformal classes which contain a product metric near the boundary (see [12], [1]). The minimal boundary condition is crucial to prove the following approximation result.

Theorem 2.2. ([1, Theorem 4.6]) *Let W be a manifold with boundary $\partial W = M$, $C \in \mathcal{C}(M)$. Let $\bar{g} \in \mathcal{R}\mathrm{iem}^0_C(W,M)$ be a metric. Let $g = \bar{g}|_M$, and $A_{\bar{g}}$ be the second fundamental form of $M = \partial W$. There exists a family of metrics $\tilde{g}_\delta \in \mathcal{R}\mathrm{iem}^0_C(W,M)$ such that*

- **(i)** *$\tilde{g}_\delta \to \bar{g}$ in the C^0-topology on W (as $\delta \to 0$),*
- **(ii)** *$R_{\tilde{g}_\delta} \to R_{\bar{g}}$ in the C^0-topology on W (as $\delta \to 0$),*
- **(iii)** *$\tilde{g}_\delta$ conformally equivalent to the metric $g + dr^2$ on $U_{\varepsilon(\delta)}(M, \bar{g})$,*
- **(iv)** *$\tilde{g}_\delta \equiv \bar{g}$ on $W \setminus U_\delta(M, \bar{g})$.*

In terms of the relative Yamabe constant, Theorem 2.2 gives the following conclusion:

Corollary 2.3. ([1, Theorem 2.1]) *For any $\bar{C} \in \mathcal{C}_C(W,M)$, and any $\varepsilon > 0$ there exist a conformal class $\tilde{C} \in \mathcal{C}_C(W,M)$, and a metric $\tilde{g} \in \tilde{C}^0$, such that*

$$\left\{\begin{array}{l} \bar{C} \text{ and } \tilde{C} \text{ are } C^0\text{-close conformal classes} \\ |Y_{\bar{C}}(W,M;C) - Y_{\tilde{C}}(W,M;C)| < \varepsilon \\ \tilde{g} \sim g + dr^2 \ \ \text{(conformally equivalent near } M\text{)}, \end{array}\right. \tag{2}$$

where $C = \partial\tilde{C}$ and $g = \bar{g}|_M$.

2.5. Conformal cobordism. We call a conformal class $C \in \mathcal{C}(M)$ *positive* if the Yamabe constant $Y_C(M) > 0$. Let $\mathcal{C}^+(M) \subset \mathcal{C}(M)$ be the space of all positive conformal classes. A pair (M, C) with $C \in \mathcal{C}^+(M)$ is called a *positive conformal manifold*. Recall that positive conformal manifolds (M_0, C_0), (M_1, C_1) are *conformally cobordant* if there exists a smooth cobordism W with the boundary $\partial W = M_0 \sqcup (-M_1)$, and such that the relative Yamabe invariant $Y(W, M_0 \sqcup M_1; C_0 \sqcup C_1) > 0$. We proved in [1] that the conformal cobordism is an equivalence relation.

2.5. Cylindrical manifolds. It is convenient for us to use a general concept of cylindrical manifolds. Let Z be a compact, closed smooth manifold, $\dim Z = n-1$. In general, Z may have several connective components; we let

$$Z = \bigsqcup_{j=1}^{m} Z_j, \quad \text{where each } Z_j \text{ is connected.}$$

Let $\mathcal{R}\mathrm{iem}(Z)$ be the space of Riemannian metrics on the manifold Z. We let $h \in \mathcal{R}\mathrm{iem}(Z)$ to be fixed.

Definition 2.1. Let $(X, \bar{g})$ be a complete Riemannian manifold, $\dim X = n$. We call $(X, \bar{g})$ a *cylindrical manifold modeled by* (Z, h) if there exists a compact smooth manifold W with non-empty boundary $\partial W = Z \sqcup M$ such that

$$\left\{\begin{array}{ll} X \overset{\text{diff}}{\cong} W \cup_Z (Z \times [0,\infty)) & \text{where } \partial W \supset Z \text{ is identified with } Z \times \{0\} \subset Z \times [0,\infty), \\ & \\ \bar{g}(z,t) = h(z) + dt^2 & \text{on } Z \times [1,\infty) \text{ with } (z,t) \text{ coordinates on } Z \times [1,\infty) \end{array}\right.$$

(see Fig. 2.1). The metric $\bar{g}$ is called a *cylindrical metric on X*.

We define the space of cylindrical metrics on X:

$$\mathcal{R}\mathrm{iem}^{\mathrm{cyl}}(X) = \{\bar{g} \in \mathcal{R}\mathrm{iem}(X) \mid \bar{g} \text{ is cylidrical as in Definition 2.1}\}$$

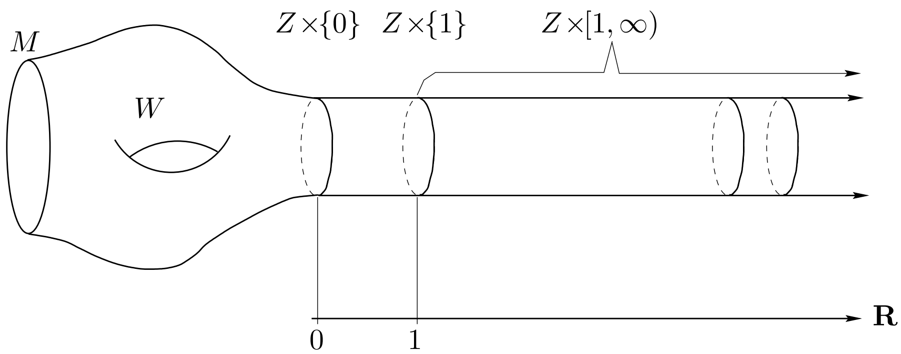

Fig. 2.1. A cylindrical manifold X

We define the space of *cylindrical conformal classes* on X as

$$\mathcal{C}^{\mathrm{cyl}}(X) = \left\{[\bar{g}] \mid \bar{g} \in \mathcal{R}\mathrm{iem}^{\mathrm{cyl}}(X)\right\}.$$

Remark. The category of cylindrical manifolds is well-suited for the conformal geometry. In particular, there are well-defined *cylindrical* Yamabe constant and Yamabe invariant. The authors plan to explore this in different paper.

3 Main Theorem: outline of the proof

3.1. Setting. Let W be a compact smooth manifold with $\dim W = n \geq 3$, and $\partial W = Z \sqcup M \neq \emptyset$. Let $C \in \mathcal{C}^+(Z)$ be a positive conformal class, and $C' \in \mathcal{C}(M)$ a conformal class. (Here C' may not be positive, in general.) We choose a metric $h \in C \sqcup C'$ with $R_h > 0$. We assume that it is given a conformal class $\tilde{C} \in \mathcal{C}(W)$ with $\partial\tilde{C} = C \sqcup C'$, and that $Y_{\tilde{C}}(W, Z \sqcup M; C \sqcup C') > 0$. We use Theorem 2.2 and [1, Theorem 5.1] to choose a conformal class

$$\bar{C} \in \mathcal{C}(W \cup_Z (Z \times [0,1])) = \mathcal{C}(X(1))$$

on the manifold $X(1) = W \cup_Z (Z \times [0,1])$ satisfying the following properties:

- **(1)** the restriction $\bar{C}|_W$ is a small pertubation of $\tilde{C}$ near $Z \times \{0\}$;
- **(2)** there is a metric $\bar{g} \in \bar{C}$ such that $\bar{g} = h + dt^2$ near $Z \times \{1\}$;
- **(3)** $Y_{\bar{C}}(X(1), Z \sqcup M; C \sqcup C') > 0$.

We extend "cylindrically" the metric $\bar{g}$ and the conformal class $\bar{C} \in \mathcal{C}(X(1))$ to the cylindrical manifold

$$X = W \cup_Z (Z \times [0,\infty)) = X(1) \cup_Z (Z \times [1,\infty)).$$

We denote those extensions also $\bar{g} \in \mathcal{R}\mathrm{iem}^{\mathrm{cyl}}(X)$, and $\bar{C} \in \mathcal{C}^{\mathrm{cyl}}(X)$. The resulting manifold X is a cylindrical manifold modeled by (Z, h). Thus we have

$$\begin{cases} Y_{\bar{C}|_{X(1)}}(X(1), Z \sqcup M; C \sqcup C') > 0, \\ \bar{g} = h + dt^2 \quad \text{on } Z \times (1-\varepsilon, \infty) \text{ for some } \varepsilon > 0. \end{cases}$$

We define $X(\ell) = W \cup_Z (Z \times [0,\ell])$ for $\ell \geq 1$.

The following theorem is the main technical result in this section. We use the above notations in this theorem.

Theorem 3.1. *Let W be a compact smooth manifold, $\dim W \geq 3$, with $\partial W = Z \sqcup M$, $Z \neq \emptyset$. Let $C \in \mathcal{C}^+(Z)$, $C' \in \mathcal{C}(M)$, and let $h \in C$ be a given metric with $R_h > 0$.*

Let $\tilde{C} \in \mathcal{C}(W)$ be a conformal class with $\partial \tilde{C} = C \sqcup C'$, such that the relative Yamabe constant $Y_{\tilde{C}}(W, Z \sqcup M; C \sqcup C') > 0$. Let X be the above cylindrical manifold modeled by (Z,h).

Then there exist a constant $L >> 1$, a conformal class $\bar{C} \in \mathcal{C}^{\mathrm{cyl}}(X(L))$ with $\partial \bar{C} = C \sqcup C'$, and a metric $\hat{g} \in \bar{C}^0$, such that

$$\begin{cases} R_{\hat{g}} > 0 & \text{on } X(L), \\ \hat{g} = h + dt^2 & \text{on } Z \times [L-1, L]. \end{cases} \tag{3}$$

3.2. Outline of the proof of Theorem 3.1. From now on, for simplicity we assume that $M = \emptyset$, that is $\partial W = Z$ (see Fig. 4.1). The proof of the case $M \neq \emptyset$ is rather similar to the one given below. To make our first steps we observe the following.

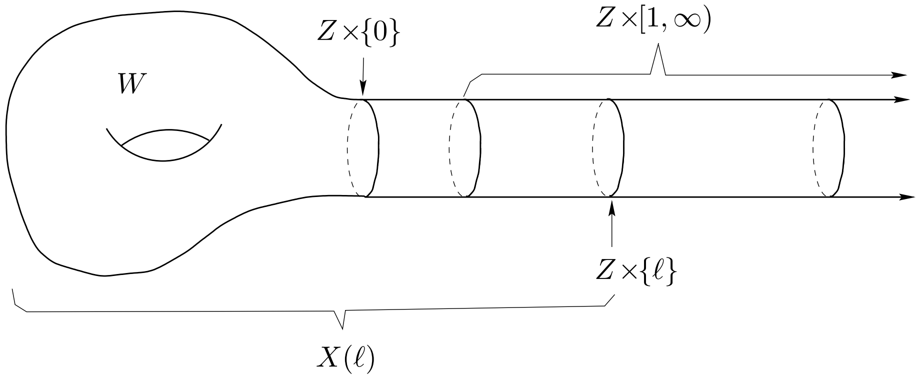

Fig. 4.1. The cylindrical manifold X modeled by (Z,h)

Observation. We may assume that the metric h on Z is a Yamabe metric with $R_h \equiv 1$. Indeed, we did not impose any conditions on the volume of Z, and if there is any psc-metric $h' \in [h]$, then the metrics h and h' are isotopic (and, consequently, are concordant). This follows from the fact that the set of psc-metrics $P(C) \subset C$ is convex for any positive conformal class $C \in \mathcal{C}^+(Z)$ (see [1, Lemma 7.2]).

Now we start with the conformal manifold $(W, \tilde{C})$ and construct the cylindrical manifold X modeled by (Z,h) as above. In particular, we choose a conformal class $\bar{C} \in \mathcal{C}^{\mathrm{cyl}}(X)$, with $\bar{g} \in \bar{C}$ as it was described. The idea is to construct a function $v_\ell \in C^\infty(X)$, so that the conformal metric $\hat{g}_\ell = v_\ell^{\frac{4}{n-2}} \bar{g}$ on the cylindrical manifold X satisfies the conditions

$$\begin{cases} R_{\hat{g}_\ell} > 0 & \text{on } X(\ell+2), \\ \hat{g}_\ell = h + dt^2 & \text{on } Z \times [\ell+1, \ell+2] \end{cases} \tag{4}$$

for some $\ell >> 1$. We achieve this in three steps.

Step 1. We study the Yamabe operator

$$L_{\bar{g}} = -\frac{4(n-1)}{n-2}\Delta_{\bar{g}} + R_{\bar{g}}$$

on the manifold $X(\ell)$ for $\ell \geq 1$. Namely, we study the linear equation

$$L_{\bar{g}} u = \lambda_\ell u \quad \text{on } X(\ell),\ u|_{\partial X(\ell)} \equiv 0, \tag{5}$$

with the Dirichlet boundary condition, where λ_ℓ is the corresponding first eigenvalue of $L_{\bar{g}}$ (for the Dirichlet boundary problem). For each $\ell \geq 1$ we find a function u_ℓ satisfying (5) and the conditions

$$u_\ell > 0 \ \text{ on the interior of } X(\ell), \text{ and with } \quad \min_{Z\times\{1\}} u_\ell = 1.$$

In order to control the first eigenvalue λ_ℓ, we define the invariant $\nu_1 = \nu_1(\bar{g})$ (see formula (6) below) which is not a conformal invariant. However we note (Claim 4.1) that the positivity of $Y_{\bar{C}}(X(1), Z; C)$ implies positivity of $\nu_1(\tilde{g})$ for any metric $\tilde{g} \in \bar{C}^0$. Then we change conformally the metric $\bar{g}$ within the interior of $X(1)$ to achieve the bound $\nu_1(\bar{g}) \geq 1$.

Step 2. We show that for the resulting metric $\bar{g}$ the eigenvalues λ_ℓ are bounded from below: $\lambda_\ell \geq \nu_1 \geq 1$. Then we prove several estimates (Claims 4.4, 4.5, 4.9, 4.8) on the eigenfunction u_ℓ. It is important that these estimates are independent of ℓ.

Step 3. Here we choose a cut-off function ϕ_ℓ to define $v_\ell = (1-\phi_\ell)u_{\ell+1} + \phi_\ell$ and examine the scalar curvature of the conformal metric $\hat{g}$. We show that $\hat{g}$, indeed, satisfies the conditions (4) for some $\ell >> 1$.

4 Proof of Theorem 3.1

Step 1. We define the invariant $\nu_1 = \nu_1(\bar{g})$ as follows

$$\nu_1 = \nu_1(\bar{g}) = \inf_{\substack{f\in C^\infty(X(1))\\ f\not\equiv 0}} \frac{\int_{X(1)} \left[\frac{4(n-1)}{n-2}|df|^2_{\bar{g}} + R_{\bar{g}} f^2\right] d\sigma_{\bar{g}}}{\int_{X(1)} f^2 d\sigma_{\bar{g}}}. \tag{6}$$

One observes the following implication.

Claim 4.1. *If $Y_{\bar{C}|_{X(1)}}(X(1), Z; C) > 0$ then $\nu_1(\tilde{g}) > 0$ for any metric $\tilde{g} \in \bar{C}^0$.*

Thus the condition $\nu_1(\tilde{g}) > 0$ is conformally invariant.

Claim 4.2. *Under above conditions there exists a metric $\breve{g} \in \bar{C}|_{X(1)}$ satisfying*

$$\begin{cases} \breve{g} \equiv \bar{g} & \text{on } Z\times[1-\varepsilon, 1] \\ \nu_1(\breve{g}) \geq 1 \end{cases}$$

for some $\varepsilon > 0$.

Proof. We choose $\breve{g} \in \bar{C}|_{X(1)}$ (keeping the condition $\breve{g} = d + dt^2$ near $Z \times \{1\}$) so that $X(1)$ has a small volume $\delta = \mathrm{Vol}_{\breve{g}}(X(1))$ (see Fig. 4.2). By Hölder inequality, we have

$$\int_{X(1)} f^2 d\sigma_{\breve{g}} \leq \mathrm{Vol}_{\breve{g}}(X(1))^{\frac{2}{n}} \cdot \left(\int_{X(1)} |f|^{\frac{2n}{n-2}} d\sigma_{\breve{g}} \right)^{\frac{n-2}{n}}$$

for any $f \in C^\infty(X(1))$. This implies

$$\frac{1}{\int_{X(1)} f^2 d\sigma_{\breve{g}}} \geq \frac{1}{\delta^{\frac{2}{n}}} \cdot \frac{1}{\left(\int_{X(1)} |f|^{\frac{2n}{n-2}} d\sigma_{\breve{g}} \right)^{\frac{n-2}{n}}}, \quad \text{and then}$$

$$\frac{\int_{X(1)} \left[\frac{4(n-1)}{n-2} |df|^2_{\breve{g}} + R_{\breve{g}} f^2 \right] d\sigma_{\breve{g}}}{\int_{X(1)} f^2 d\sigma_{\breve{g}}} \geq \frac{1}{\delta^{\frac{2}{n}}} \cdot \frac{\int_{X(1)} \left[\frac{4(n-1)}{n-2} |df|^2_{\breve{g}} + R_{\breve{g}} f^2 \right] d\sigma_{\breve{g}}}{\left(\int_{X(1)} |f|^{\frac{2n}{n-2}} d\sigma_{\breve{g}} \right)^{\frac{n-2}{n}}}.$$

Thus we obtain that

$$\nu_1(\breve{g}) \geq \frac{1}{\delta^{\frac{2}{n}}} \cdot Y_{\bar{C}|_{X(1)}}(X(1), Z; [h]),$$

where $Y_{\bar{C}|_{X(1)}}(X(1), Z; [h])$, perhaps, is a conformal invariant. Finally we choose δ small enough to complete the proof of Claim 4.2. □

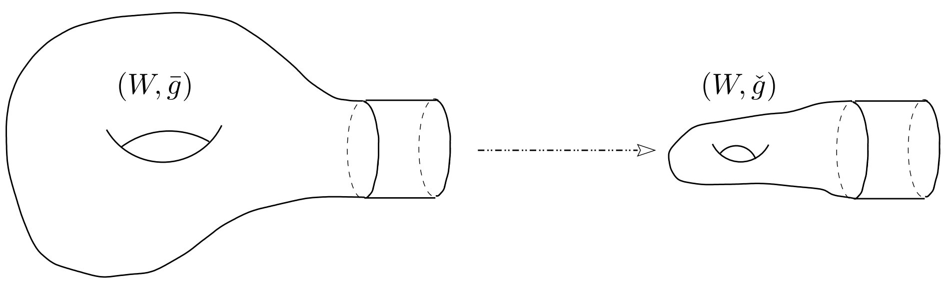

Fig. 4.2. Metric $\breve{g}$ on $X(1)$.

For simplicity, we denote $\breve{g}$ by $\bar{g}$. We summarize the properties of the metric $\bar{g} \in \bar{C}$ on $X = W \cup_Z (Z \times [0, \infty))$:

$$\begin{cases} \nu_1(\bar{g}) \geq 1 \equiv R_h, \\ \bar{g} = h + dt^2 \quad \text{on } Z \times [1 - \varepsilon, \infty). \end{cases} \tag{7}$$

Step 2. For any $\ell \geq 1$ we define

$$\lambda_\ell = \lambda_\ell(\bar{g}) = \inf_{\substack{f \in C^\infty(X(\ell)) \\ f|_{\partial X(\ell)} \equiv 0, \quad f \not\equiv 0}} \frac{\int_{X(\ell)} \left[\frac{4(n-1)}{n-2} |df|^2_{\bar{g}} + R_{\bar{g}} f^2 \right] d\sigma_{\bar{g}}}{\int_{X(\ell)} f^2 d\sigma_{\bar{g}}}.$$

One easily proves the following statement.

Claim 4.3. *The numbers λ_ℓ satisfy*

$$\begin{cases} \lambda_\ell \geq \min\{\nu_1, R_h\} \quad (\textit{where } R_h \equiv 1) \textit{ for } \ell \geq 1, \\ \lambda_1 \geq \lambda_{\ell_1} \geq \lambda_{\ell_2} \geq 1 \quad \textit{for } 1 < \ell_1 < \ell_2. \end{cases}$$

In particular, we have

$$\lambda_1 \geq \lambda_\ell \geq R_h \equiv 1 \quad \text{for } \ell \geq 1. \tag{8}$$

Now let $\ell > 1$. Then there exists a function $u_\ell \in C^\infty(X(\ell))$ such that

$$\begin{cases} L_{\bar{g}} u_\ell = -\dfrac{4(n-1)}{n-2}\Delta_{\bar{g}} u_\ell + R_{\bar{g}} u_\ell = \lambda_\ell u_\ell \quad \text{on } X(\ell), \\ u_\ell > 0 \qquad \text{on the interior of } X(\ell), \\ u_\ell|_{\partial X(\ell)} \equiv 0 \quad \text{(Dirichlet boundary condition)}, \\ \min\limits_{Z \times \{1\}} u_\ell = 1 \quad \text{(normalization condition)}. \end{cases}$$

We define a function $\psi_\ell \in C^\infty(Z \times [1,\ell])$ by $\psi_\ell(z,t) = 1 - \dfrac{t-1}{\ell-1}$ (see Fig. 4.3).

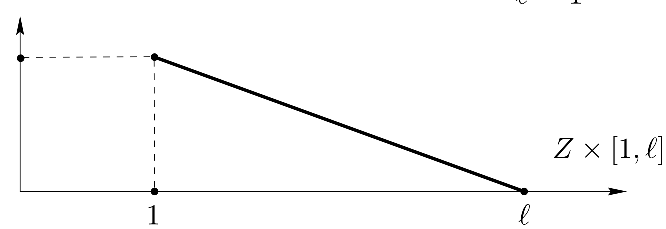

Fig. 4.3. Function ψ_ℓ

Clearly $\Delta_{\bar{g}}\psi_\ell = \partial_t^2 \psi_\ell \equiv 0$ on $Z \times [1,\ell]$. We observe the following fact.

Claim 4.4. *The function u_ℓ satisfies the inequality $u_\ell \geq \psi_\ell$ on $Z \times [1,\ell]$ for any $\ell > 1$.*

Proof. The above condition on u_ℓ gives that $L_{\bar{g}} u_\ell = \lambda_\ell u_\ell$. This implies

$$\begin{aligned} \Delta_{\bar{g}} u_\ell &= \frac{n-2}{4(n-1)}(R_{\bar{g}} - \lambda_\ell) u_\ell \\ &= \frac{n-2}{4(n-1)}(R_h - \lambda_\ell) u_\ell \leq 0 \quad \text{on } Z \times [1,\ell] \end{aligned} \tag{9}$$

since $u_\ell \geq 0$ and by (7). We obtain that $\Delta_{\bar{g}}(\psi_\ell - u_\ell) \geq 0$. Thus by the maximum principle,

$$\psi_\ell - u_\ell \leq \max_{\partial(Z \times [1,\ell])} (\psi_\ell - u_\ell) = 0$$

because of the choice of ψ_ℓ (here, of course, $\partial(Z \times [1,\ell]) = (Z \times \{1\}) \sqcup (Z \times \{\ell\})$). Then we obtain that $u_\ell \geq \psi_\ell$ on $Z \times [1,\ell]$. □

Claim 4.5. *Let $\ell > 3$. Then there exists a constant $K > 0$ independent of ℓ, so that*

$$|du_\ell| \leq K \cdot |u_\ell| \quad \text{on } Z \times [1, \ell - 2].$$

Proof. Let $x \in X$. We define $B_1(x) = \{ y \in X \mid \mathrm{dist}_{\bar{g}}(y, x) \leq 1 \}$. Now we have to recall the following facts.

Fact 4.6. (see [10])
Let $f \in C^\infty(X)$. Then for any $x \in X$ there exists a constant $K_1 > 0$ so that

$$|df(x)| \leq K_1 \left[\int_{B_1(x)} |\Delta_{\bar{g}} f| \, d\sigma_{\bar{g}} + \int_{B_1(x)} |f| \, d\sigma_{\bar{g}} + \int_{\partial B_1(x)} |f| \, d\sigma_{\bar{g}|_{\partial B_1(x)}} \right].$$

Fact 4.7. (Harnack inequality, see [7, Theorem 8.20])
There exists a constant $K_2 > 0$ independent of $\ell > 1$ so that

$$|u_\ell| \leq K_2 \cdot \left(\inf_{y \in B_1(x)} u_\ell(y) \right) \qquad \text{for } x \in Z \times [1, \ell - 1].$$

We continue with the proof of Claim 4.5. The Facts 4.6, 4.7, (8) and (9) imply that there exists a constant $K_3 > 0$ so that

$$\begin{aligned} |du_\ell(x)| &\leq K_1 \cdot \left[K_3 \int_{B_1(x)} |u_\ell| d\sigma_{\bar{g}} + \int_{B_1(x)} |u_\ell| d\sigma_{\bar{g}} + \int_{\partial B_1(x)} |u_\ell| \, d\sigma_{\bar{g}|_{\partial B_1(x)}} \right] \\ &\leq K_1 K_2 (K_3 + 2) \cdot \left(\inf_{y \in B_1(x)} u_\ell(y) \right) \end{aligned}$$

for $x \in Z \times [1, \ell - 2]$. This completes the proof of Claim 4.5. □

Recall that $\min\limits_{Z \times \{1\}} u_\ell = 1$. Now Claim 4.5 implies that there exists a constant $\bar{K} > 0$, independent of ℓ, so that $\max\limits_{Z \times \{1\}} u_\ell \leq \bar{K}$.

Claim 4.8. *The function u_ℓ satisfies*

$$u_\ell \leq \bar{K} \quad \text{on } Z \times [1, \ell] \text{ for any } \ell > 1.$$

Proof. Consider the function $e^{\delta t} \cdot u_\ell$ for $\delta > 0$. Recall that

$$\Delta_{\bar{g}} u_\ell = \frac{n-2}{4(n-1)} (R_h - \lambda_\ell).$$

We use (8) and Claim 4.5 to see the following estimate:

$$\begin{aligned} \Delta_{\bar{g}} (e^{\delta t} \cdot u_\ell) &= e^{\delta t} \cdot \Delta_{\bar{g}} u_\ell + 2\delta \cdot e^{\delta t} u_\ell' + \delta^2 \cdot e^{\delta t} u_\ell \\ &\geq -\frac{n-2}{4(n-1)} \lambda_1 \cdot e^{\delta t} u_\ell - 2\delta K e^{\delta t} u_\ell + \delta^2 \cdot e^{\delta t} u_\ell \\ &\geq \left(\delta^2 - \frac{n-2}{4(n-1)} \lambda_1 - 2\delta K \right) e^{\delta t} u_\ell \geq 0 \end{aligned}$$

for large enough $\delta >> 1$ (since $u_\ell \geq 0$), where $(\cdot)' = \frac{\partial}{\partial t}(\cdot)$. Now the maximum principle gives

$$e^{\delta t} u_\ell \leq \max_{\partial(Z\times[1,\ell])} e^{\delta t} u_\ell = \max_{Z\times\{1\}} e^{\delta t} u_\ell \leq e^{\delta t} \cdot \bar{K}$$

on $Z \times [1, \ell]$ since $u_\ell|_{Z\times\{\ell\}} = 0$. Thus we obtain $u_\ell(z, t) \leq e^{\delta(1-t)} \cdot \bar{K} \leq \bar{K}$ on $Z \times [1, \ell]$. □

We need one more precise estimate on the function u_ℓ.

Claim 4.9. *There exist constants $\tilde{K}_1 > 0$, $\tilde{K}_2 > 0$ (independent of ℓ) such that*

$$|du_\ell|_{C^{0,\alpha}} \leq \tilde{K}_1 \left(|u_\ell| + \tilde{K}_2\right) \quad \text{on } Z \times [1, \ell].$$

Proof. Indeed, we have that the function u_ℓ satisfies

$$\begin{cases} u_\ell|_{Z\times\{\ell\}} \equiv 0, \\ 0 \leq u_\ell \leq \bar{K} \quad \text{on } Z \times [1, \ell], \\ |du_\ell| \leq K \cdot |u_\ell| \quad \text{on } Z \times [1, \ell - 2]. \end{cases}$$

Recall that we have

$$\Delta_{\bar{g}} u_\ell = \frac{n-2}{4(n-1)}(R_{\bar{g}} - \lambda_\ell) \quad \text{on } Z \times [1, \ell]$$

$$\lambda_1 \geq \lambda_\ell \geq R_h \equiv 1.$$

Then, by standard argument, we obtain that

$$|du_\ell|_{C^{0,\alpha}} \leq \tilde{K} \cdot |u_\ell| \quad \text{on } Z \times [1, \ell - 2].$$

Then [7, Theorem 8.33] implies that there exist constants $\tilde{K}_1 > 0$, $\tilde{K}_2 > 0$ such that

$$|du_\ell|_{C^{0,\alpha}} \leq \tilde{K}_1 \left(|u_\ell| + \tilde{K}_2\right) \quad \text{on } Z \times [1, \ell].$$

This completes the proof of Claim 4.9. □

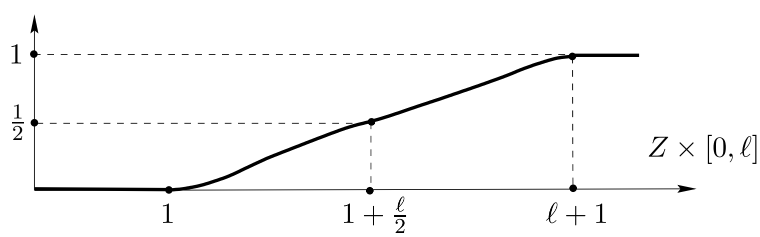

Fig. 4.4. Function ϕ_ℓ

Step 3. Let $\ell >> 1$. Let $\phi_\ell \in C^\infty(X)$ be a cut-off function satisfying the following conditions (see Fig. 4.4):

(1) $\phi_\ell(x) = \begin{cases} 0 & \text{for } x \in X(1), \\ 1 & \text{for } x \in Z \times [\ell + 1, \infty) \end{cases}$

(2) $0 \leq \phi_\ell \leq 1$ on X, and $\phi_\ell(z, t) = \phi_\ell(t)$ for $(z, t) \in Z \times [1, \ell + 1]$.

(3) For some constant $\hat{K} > 0$ (independent of ℓ)

$$0 \le \phi'_\ell \le \frac{\hat{K}}{\ell}, \quad |\phi''_\ell| \le \frac{\hat{K}}{\ell^2} \quad \text{on } Z \times [1, \ell+1].$$

(4) Moreover, $\phi_\ell(1 + \frac{\ell}{2}) = \frac{1}{2}$ on $Z \times \left\{1 + \frac{\ell}{2}\right\}$.

It is not difficult to find such function ϕ_ℓ. We let $v_\ell = (1 - \phi_\ell) \cdot u_{\ell+1} + \phi_\ell \in C^\infty_+(X)$, and the conformal metric $\hat{g}_\ell = v_\ell^{\frac{4}{n-2}} \cdot \bar{g}$ on X. Then the scalar curvature of the metric $\hat{g}_\ell$ is given by

$$R_{\hat{g}_\ell} = v_\ell^{-\frac{n+2}{n-2}} \left[-\frac{4(n-1)}{n-2} \Delta_{\bar{g}} v_\ell + R_{\bar{g}} v_\ell \right] = v_\ell^{-\frac{n+2}{n-2}} L_{\bar{g}} v_\ell.$$

We examine the scalar curvature $R_{\hat{g}_\ell}$ on three different pieces:

$$X = X(1) \cup (Z \times [1, \ell+1]) \cup (Z \times [\ell+1, \infty)).$$

• The piece $X(1)$. Then we have that $v_\ell \equiv u_{\ell+1}$, thus

$$R_{\hat{g}_\ell}|_{X(1)} = u_{\ell+1}^{-\frac{n+2}{n-2}} \cdot (\lambda_{\ell+1} u_{\ell+1}) = \lambda_{\ell+1} \cdot u_{\ell+1}^{-\frac{4}{n-2}} > 0.$$

• The piece $Z \times [\ell+1, \infty)$. Here we have that $v_\ell \equiv 1$ (which is equivalent to the fact that $\hat{g}_\ell = \bar{g} = h + dt^2$). Thus

$$R_{\hat{g}_\ell}|_{Z \times [\ell+1, \infty)} = R_{\bar{g}} = R_h \equiv 1.$$

• The piece $Z \times [1, \ell+1]$. This case is more complicated. We have:

$$\begin{aligned} R_{\hat{g}_\ell}|_{Z \times [1, \ell+1]} &= v_\ell^{-\frac{n+2}{n-2}} \left[(1 - \phi_\ell) L_{\bar{g}} u_{\ell+1} + \frac{4(n-1)}{n-2} (u_{\ell+1} - 1) \phi''_\ell \right. \\ &\qquad \left. + \frac{8(n-1)}{n-2} \phi'_\ell \cdot u'_{\ell+1} + R_h \phi_\ell \right] \\ &= v_\ell^{-\frac{n+2}{n-2}} \left[(1 - \phi_\ell) \lambda_{\ell+1} \cdot u_{\ell+1} + \frac{4(n-1)}{n-2} (u_{\ell+1} - 1) \phi''_\ell \right. \\ &\qquad \left. + \frac{8(n-1)}{n-2} \phi'_\ell \cdot u'_{\ell+1} + \phi_\ell \right]. \end{aligned}$$

We use (8), Claim 4.8, Claim 4.9, and property (3) of the function ϕ_ℓ to get the estimation:

$$R_{\hat{g}_\ell}|_{Z \times [1, \ell+1]} \ge v_\ell^{-\frac{n+2}{n-2}} \left[(1 - \phi_\ell) u_{\ell+1} - \frac{8 \bar{K} \hat{K}}{\ell^2} - \frac{8 \bar{K} \tilde{K}_1 (\hat{K} + \tilde{K}_2)}{\ell} + \phi_\ell \right].$$

Now we examine even more carefully the scalar curvature $R_{\hat{g}_\ell}$ on the cylinder

$$Z \times [1, \ell+1] = \left(Z \times [1, 1 + \tfrac{\ell}{2}]\right) \cup \left(Z \times [1 + \tfrac{\ell}{2}, \ell+1]\right).$$

• The piece $Z \times [1, 1+\frac{\ell}{2}]$. Here the property (4) of the function ϕ_ℓ and Claim 4.4 imply:

$$\left\{ \begin{array}{l} 1-\phi_\ell \geq \frac{1}{2}, \\ u_{\ell+1} \geq \psi_{\ell+1} \geq \psi_{\ell+1}(1+\frac{\ell}{2}) \geq \frac{1}{2}. \end{array} \right.$$

Thus we have that

$$R_{\hat{g}_\ell}|_{Z\times[1,1+\frac{\ell}{2}]} \geq v_\ell^{-\frac{n+2}{n-2}} \left(\frac{1}{4} - \frac{8\bar{K}\hat{K}}{\ell^2} - \frac{8\bar{K}\tilde{K}_1(\hat{K}+\tilde{K}_2)}{\ell} \right).$$

Clearly there exists such $\ell_1 >> 1$ that $R_{\hat{g}_\ell} > 0$ on $Z \times [1, 1+\frac{\ell}{2}]$ for all $\ell \geq \ell_1$.

• The piece $Z \times [1+\frac{\ell}{2}, \ell+1]$. Here we have $\phi_\ell \geq \frac{1}{2}$ by the conditions (3) and (4) on the function ϕ_ℓ. Thus we we have

$$R_{\hat{g}_\ell}|_{Z\times[1+\frac{\ell}{2},\ell+1]} \geq v_\ell^{-\frac{n+2}{n-2}} \left(\frac{1}{2} - \frac{8\bar{K}\hat{K}}{\ell^2} - \frac{8\bar{K}\tilde{K}_1(\hat{K}+\tilde{K}_2)}{\ell} \right).$$

Thus there exists $\ell_2 >> 1$ such that $R_{\hat{g}_\ell} > 0$ on $Z \times [1+\frac{\ell}{2}, \ell+1]$ for all $\ell \geq \ell_2$.

Now let $\ell_0 = \max\{\ell_1, \ell_2\}$, and let $\hat{g} = \hat{g}_{\ell_0}$, and $L = \ell_0 + 2$. Thus we constructed a metric $\hat{g} \in \mathcal{R}\mathrm{iem}(X(L))$ such that

$$\left\{ \begin{array}{ll} R_{\hat{g}} > 0 & \text{on } X(L), \\ \hat{g} = h + dt^2 & \text{on } Z \times [L-1, L]. \end{array} \right.$$

This completes the proof of Theorem 3.1. □

5 Some topology

5.1. Summary on γ-structures. We briefly review necessary definitions and constructions given by S. Stolz [16]. Let $\pi = \pi_1(M)$ be the fundamental group of M, and $w_i(M) \in H^i(M; \mathbf{Z}_2)$ be the Stiefel-Whitney characteristic classes.

The main conceptual issue here is to determine precisely which topological structure on a smooth compact manifold M carries complete information on the existence of a psc-metric on M. Indeed, it is well-known that the fundamental group π is crucially important for the existence question. Then there is clear difference when a manifold M is oriented or not (which depends on $w_1(M)$). On the other hand, a presence of the *Spin*-structure (which means that $w_2(M) = 0$) gives a way to use the Dirac operator on M to control the scalar curvature via the vanishing formulas. S. Stolz puts together those invariants to define a γ-structure.

To simplify our presentation, we consider only the case of oriented manifolds. In the oriented case the γ-structures have very transparent geometric description. The non-oriented case is more subtle and complicated; we would like to live this case outside of our paper. However, in our view, one should not meet any difficulties to generalize our results to the non-oriented case.

Let M be an oriented manifold with $\pi = \pi_1(M)$. Let $f : M \longrightarrow B\pi$ be a classifying map for the fundamental group, and $p : \widetilde{M} \to M$ be the universal cover. Recall that the second Stiefel-Whitney class $w_2 = w_2(M)$ is zero if and only if the manifold M admits a *Spin* structure. We have the following three cases to consider:

(1) $w_2 = 0$, thus the manifold M is a *Spin* manifold;

(2) $w_2 \neq 0$, but the universal cover $\widetilde{M}$ is a *Spin* manifold;

(3) $w_2 \neq 0$, and the universal cover $\widetilde{M}$ is not a *Spin* manifold.

Comments. **(1)** In this case M admits a *Spin* structure, however it is important to choose the *Spin* structure. We call a manifold M a *Spin*-manifold if the *Spin*-structure is chosen. A classifying map $f : M \longrightarrow B\pi$ then determines a canonical cobordism class $[(M, f)] \in \Omega_n^{\mathrm{Spin}}(B\pi)$. In this case a γ-structure on M is defined as a choosen *Spin*-structure on M together with the classifying map $f : M \longrightarrow B\pi$ for the fundamental group π.

(2) This case involves more. Consider the induced homomorphism

$$f^* : H^2(B\pi; \mathbf{Z}_2) \to H^2(M; \mathbf{Z}_2).$$

In this case S. Stolz proves that there exists a unique element $e \in H^2(B\pi; \mathbf{Z}_2)$, so that $f^*(e) = w_2$. The element e, as any element of $H^2(B\pi; \mathbf{Z}_2)$, determines a central group extension $1 \to \mathbf{Z}_2 \to \hat{\pi} \xrightarrow{\rho} \pi \to 1$. Futhermore, this extension splits (or trivial) if and only if $e = 0$. Thus the pair $(\hat{\pi}, \pi)$ completely encodes the case (2) and the case (1) as well (then $e = 0$, and $\hat{\pi} \cong \pi \times \mathbf{Z}_2$). This gives the γ-structure $\gamma = (\hat{\pi}, 0, \pi)$ in the notations of [16]. Alternatively, this structure gives the following construction. Let $\sigma \in \mathbf{Z}_2 \subset \hat{\pi}$ be a generator. Then the element $(\sigma, -1)$ is central in the direct product $\hat{\pi} \times Spin(n)$. The Lie group $G(\gamma, n)$ is defined as a factor group of $\hat{\pi} \times Spin(n)$ by the central subgroup $\mathbf{Z}_2$ (generated by $(\sigma, -1)$). By construction, the group $G(\gamma, n)$ has a canonical homomorphism $j : G(\gamma, n) \longrightarrow SO(n)$.

Now let g be a Riemannian metric on M, then a chosen orientation on M gives the frame bundle $P_{SO(n)}(M) \to M$. S. Stolz shows [16] that in this case the γ-structure determines a canonical principal bundle $P_{G(\gamma,n)}(M) \longrightarrow P_{SO(n)}(M)$. We obtain the principal bundle $P_{G(\gamma,n)}(M) \longrightarrow M$, and thus a map $\hat{f} : M \longrightarrow BG(\gamma, n)$ to the classifying space. The case when the above extension e is trivial gives the isomorphism $G(\gamma, n) \cong \pi \times Spin(n)$. Otherwise the group $G(\gamma, n)$ is a "twisted (by the extension e) version" of the group $Spin(n)$.

We remark that the group $G(\gamma, n)$ determines the Thom space $MG(\gamma, n)$, and thus the cobordism groups $\Omega_n(\gamma)$ given via the Thom-Pontryagin construction. In particular, the pair $(M, \hat{f})$ determines a cobordism class in $\Omega_n(\gamma)$ (where $n = \dim M$). Both cases (1) and (2) are described in [16] as $\gamma = (\hat{\pi}, 0, \pi)$ with $\hat{\pi}$ given by the above extension e.

(3) This case is easy. The γ-structure here is nothing but a choice of orientation on M together with the classifying map $f : M \longrightarrow B\pi$. Then the pair (M, f) gives a cobordism class in the oriented cobordism ring $\Omega_n^{\mathrm{SO}}(B\pi)$. In the notations of [16], $\gamma = (\pi, 0, \pi)$.

Conclusion. We emphasize that in each of the above cases we have a well-defined cobordism category $\mathfrak{M}(\gamma)$ of manifolds equipped with γ-structure. Let M_0, M_1 be two manifolds equipped with given γ-structure. A cobordism W between M_0 and M_1 in the category $\mathfrak{M}(\gamma)$ is called γ-cobordism.

5.2. Conformal and psc-cobordism groups. Now let $\mathfrak{Pos}(\gamma)$ be the following cobordism category. The objects of $\mathfrak{Pos}(\gamma)$ are the pairs (M, g), where M is a manifold with γ-structure, and g is a psc-metric on M. Manifolds (M_0, g_0), (M_1, g_1) are psc-cobordant in the category $\mathfrak{Pos}(\gamma)$ if they there is a γ-cobordism W between M_0 and M_1, where W is given a psc-metric $\bar{g}$, so that

$$R_{\bar{g}} > 0, \quad \bar{g}|_{M_j} = g_j, \quad \text{and } \bar{g} = g_j + dt^2 \text{ near the boundary } \partial W = M_0 \sqcup (-M_1) \text{ for } j = 0, 1.$$

We denote the corresponding cobordism groups $\mathcal{P}\mathrm{os}_n(\gamma)$. We emphasize that we restrict our attention to the dimensions $n \geq 5$.

The corresponding conformal cobordism category $\mathfrak{Pos}^{\mathrm{conf}}(\gamma)$ is defined similarly. The objects of $\mathfrak{Pos}^{\mathrm{conf}}(\gamma)$ are positive conformal γ-manifolds (M, C), where, as before, M is a manifold with γ-structure, and $C \in \mathcal{C}^+(M)$ is a positive conformal class. Then two positive conformal γ-manifolds (M_0, C_0), (M_1, C_1) are conformally cobordant if there exists a γ-cobordism W between M_0 and M_1, so that the relative Yamabe invariant $Y(W, M_0 \sqcup (-M_1); C_0 \sqcup C_1) > 0$.

We denote the corresponding cobordism groups $\mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$. Here we also let $n \geq 5$ (however, all definitions make sense for $n = 2, 3, 4$ as well). The fact, that the conformal cobordism is an equivalence relation is not entirely trivial (see proof in [1]). A group structure here is given by taking a disjoint union of manifolds. We have a canonical functor $\mathfrak{Pos}(\gamma) \longrightarrow \mathfrak{Pos}^{\mathrm{conf}}(\gamma)$ given by taking conformal classes of corresponding metrics, so we have natural homomorphism $\mathcal{P}\mathrm{os}_n(\gamma) \longrightarrow \mathcal{P}\mathrm{os}_n^{\mathrm{conf}}(\gamma)$. Clearly Theorem 3.1 implies Theorem A and Corollary B. Since concordance is just a particular case of cobordism, this also implies Corollary D.

5.3. Relative cobordism groups. Now we define the cobordism category $\mathfrak{R}(\gamma)$ for a given γ-structure as above. The objects of the category $\mathfrak{R}(\gamma)$ are γ-manifolds $(M, \partial M; \bar{g}, g)$, where $\bar{g}$ is a Riemannian metric on M, and g is a psc-metric on ∂M, such that

$$\bar{g} = g + dt^2 \text{ near the boundary } \partial M.$$

In particular, if M is a closed γ-manifold, and $\bar{g}$ is any Riemannian metric, then $(M, \emptyset; \bar{g}, \emptyset)$ is an object of $\mathfrak{R}(\gamma)$. Two manifolds $(M_0, \partial M_0; \bar{g}_0, g_0)$, $(M_1, \partial M_1; \bar{g}_1, g_1)$ like this are cobordant in the category $\mathfrak{R}(\gamma)$ if there exist a γ-manifold $(W, \partial W; \tilde{g}, \hat{g})$ with given decomposition of the boundary

$$\partial W = M_0 \cup_{\partial M_0} V \cup_{\partial M_1} (-M_1),$$

where $\partial V = \partial M_0 \sqcup (-\partial M_1)$, such that (see Fig. 5.1)

(a) $\hat{g}|_{\partial V} = g_0 \sqcup g_1$, with $\hat{g} = \hat{g}|_{\partial V} + dt^2$ near ∂V,
(b) $R_{\hat{g}} > 0$ on V,
(c) $\tilde{g}|_{\partial W} = \hat{g} = \bar{g}_0 \cup \hat{g}|_V \cup \bar{g}_1$, and
(d) $\tilde{g} = \hat{g} + dt^2$ near the boundary ∂W.

Here "$-M$" means the same manifold M with the choice of *opposite γ-structure* (see [16] for more details). We remark that the manifold $(V, \hat{g}|_V)$ delivers a psc-cobordism between $(\partial M_0, g_0)$ and $(\partial M_1, g_1)$ (we emphasized this by a bold line in Fig. 5.1).

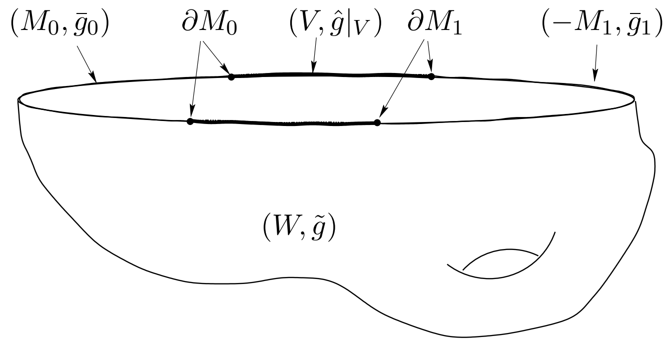

Fig. 5.1. Cobordism in the category $\mathfrak{R}(\gamma)$

Again, we emphasize that for each Riemannian manifold with boundary it is assumed here that a metric is a product metric near its boundary. Let $R_n(\gamma)$ be the corresponding cobordism groups. Disjoint union of manifolds induces an abelian group structure on $R_n(\gamma)$ (see [16]).

The conformal cobordism category $\mathfrak{R}^{\mathrm{conf}}(\gamma)$ is defined similarly. To avoid any confusions, we spell out the definition. The objects of $\mathfrak{R}^{\mathrm{conf}}(\gamma)$ are conformal γ-manifolds $(M, \partial M; \bar{C}, C)$, where $(\bar{C}, C) \in \mathcal{C}(M, \partial M)$ (i.e. $\partial \bar{C} = C \iff \bar{C}|_{\partial M} = C$) with $C \in \mathcal{C}^+(\partial M)$ positive conformal class. Conformal manifolds $(M_0, \partial M_0; \bar{C}_0, C_0)$, $(M_1, \partial M_1; \bar{C}_1, C_1)$ like this are cobordant in the category $\mathfrak{R}^{\mathrm{conf}}(\gamma)$ if there is a conformal manifold $(W, \partial W, \tilde{C}, \hat{C})$ with given decomposition of the boundary

$$\partial W = M_0 \cup_{\partial M_0} V \cup_{\partial M_1} (-M_1),$$

where $\partial V = \partial M_0 \sqcup (-\partial M_1)$, such that

$(a)^{\mathrm{conf}}$ $\hat{C}|_{\partial V} = C_0 \sqcup C_1$,
$(b)^{\mathrm{conf}}$ $Y_{\hat{C}|_V}(V, \partial M_0 \sqcup \partial M_0; C_0 \sqcup C_1) > 0$,
$(c)^{\mathrm{conf}}$ $\tilde{C}|_{\partial W} = \hat{C} = \bar{C}_0 \cup \hat{C}|_V \cup \bar{C}_1$.

Let $R_n^{\mathrm{conf}}(\gamma)$ be the corresponding cobordism groups. Clearly there are natural homomorphisms $j : \Omega_n(\gamma) \longrightarrow R_n(\gamma)$ and $j' : \Omega_n(\gamma) \longrightarrow R_n^{\mathrm{conf}}(\gamma)$, given by assigning an arbitrary Riemannian metric (or conformal class to a γ-manifold). We remark here that two closed γ-manifolds (M, g_0) and (M, g_1) (with any two metrics g_0, g_1) are cobordant in the category $\mathfrak{R}(\gamma)$ since the space of Reimannian metrics is convex. Thus a linear homotopy $g_t = (1-t)g_0 + tg_1$ gives a metric on the cylinder $M \times [0,1]$. The same is true for conformal manifolds if we do not impose any conditions on conformal classes. The maps $\partial : R_n(\gamma) \longrightarrow Pos_{n-1}(\gamma)$ and $\partial' : R_n^{\mathrm{conf}}(\gamma) \longrightarrow Pos_{n-1}^{\mathrm{conf}}(\gamma)$ are given by taking all data on boundaries. Finally one has the forgeting (metric or conformal class) homomorphisms $F : Pos_n(\gamma) \longrightarrow \Omega_n(\gamma)$, and $F' : Pos_n^{\mathrm{conf}}(\gamma) \longrightarrow \Omega_n(\gamma)$. It is easy to show that the

following diagram is commutative and has exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & R_{n+1}(\gamma) \arrow[r, "\partial"] \arrow[d, "c"] & Pos_n(\gamma) \arrow[r, "F"] \arrow[d, "\cong"] & \Omega_n(\gamma) \arrow[r, "j"] \arrow[d, "Id"] & R_n(\gamma) \arrow[r, "\partial"] \arrow[d, "c"] & Pos_{n-1}(\gamma) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & R^{\mathrm{conf}}_{n+1}(\gamma) \arrow[r, "\partial'"] & Pos^{\mathrm{conf}}_n(\gamma) \arrow[r, "F'"] & \Omega_n(\gamma) \arrow[r, "j'"] & R^{\mathrm{conf}}_n(\gamma) \arrow[r, "\partial'"] & Pos^{\mathrm{conf}}_{n-1}(\gamma) \arrow[r] & \cdots
\end{tikzcd}$$

Five-lemma implies that $c : R_n(\gamma) \longrightarrow R^{\mathrm{conf}}_n(\gamma)$ is an isomorphism. This concludes the proof of Corollary C.

6 Discussion

6.1. Concordance classes and groups $R_n(\gamma)$. To make our discussion transparent, we concentrate our attention on the case of simply connected *Spin* manifolds, then $\Omega_n(\gamma) = \Omega_n^{\mathrm{Spin}}$. In this case we omit the "γ-notation" for all cobordism groups we have here. Futhermore, we consider the simplest possible manifold, the standard sphere. Thus let $M = S^n$ for $n \geq 5$, and let Π_n be the set of psc-concordant classes of psc-metrics on S^n. Corollary D, in particular, identifies the set Π_n with its "conformal relative", the set Π_n^{conf} of conformally concordant positive conformal classes on S^n. The connective sum operation induces an abelian group structure on Π_n with zero class represented by the standard metric g_{can}. Thus Π_n^{conf} inherits this group structure.

On the other hand, it is known (see [9]) that for simply connected *Spin* manifolds the relative psc-cobordism groups R_n are naturally isomorphic to the concordance groups Π_n. We obtain the isomorphisms: $R_n^{\mathrm{conf}} \cong R_n \cong \Pi_n \cong \Pi_n^{\mathrm{conf}}$.

Conlusion. *The groups R_n^{conf} (and, consequently, the groups $\mathcal{P}\mathrm{os}_n^{\mathrm{conf}}$) are completely determined by the concordance classes of a single manifold: the sphere S^n.*

In the conformal world, we have the set $\pi_0(\mathcal{C}^+(S^n))$ of positive conformal classes.

Problem 1. *Study the relationship of the group of conformal concordance classes Π_n^{conf} and $\pi_0(\mathcal{C}^+(S^n))$.*

A study of the space $\mathcal{C}^+(S^n)$ naturally leads to an interesting model of moduli space of positive conformal classes.

6.2. Moduli spaces. Again, we consider the sphere S^n with $n \geq 5$. A standard definition of the moduli space of psc-metrics goes as follows. Let $\mathcal{R}\mathrm{iem}^+(S^n) \subset \mathcal{R}\mathrm{iem}(S^n)$ be the space of psc-metrics. The diffeomorphism group $\mathrm{Diff}_+(S^n)$ of orientation-preserving diffeomorphisms of the sphere S^n naturally acts (on the right) on the space of metrics $\mathcal{R}\mathrm{iem}(S^n)$ by pulling back a metric. Obviously this action preserves the subspace $\mathcal{R}\mathrm{iem}^+(S^n)$. There is a serious problem with this action: it is far away from to be free, leaving us very little chance to understand the topology of the moduli space $\mathcal{M}^+(S^n) = \mathcal{R}\mathrm{iem}^+(S^n)/\mathrm{Diff}_+(S^n)$ of psc-metrics.

We would like to suggest an alternative construction of such a moduli space following the paper [13] by J. Morava & H. Tamanoi. The construction below holds for arbitrary compact smooth manifold, not just for the sphere S^n.

Let $\mathcal{C}(S^n)$, $\mathcal{C}^+(S^n)$ be the spaces of all conformal classes and positive ones. The projection map $\mathcal{R}\mathrm{iem}(S^n) \longrightarrow \mathcal{C}(S^n)$ induces the map $\mathcal{R}\mathrm{iem}^+(S^n) \longrightarrow \mathcal{C}^+(S^n)$. Clearly both spaces $\mathcal{R}\mathrm{iem}(S^n)$ and $\mathcal{C}(S^n)$ are contractible, and again the diffeomorphism group $\mathrm{Diff}_+(S^n)$ action on $\mathcal{C}(S^n)$ is not free. To refine the construction, we choose a base point $x_0 \in S^n$.

The space of conformal classes $\mathcal{C}(S^n)$ is the orbit space of the action (left multiplication) of the group $C^\infty_+(S^n)$ on the space of metrics $\mathcal{R}\mathrm{iem}(S^n)$. With a given base point $x_0 \in S^n$, we consider the following subspace of $C^\infty_+(S^n)$:

$$C^\infty_{+,x_0}(S^n) = \left\{ u \in C^\infty_+(S^n) \mid u(x_0) = 1 \right\}.$$

Then let $\mathcal{C}_{x_0}(S^n)$ be the orbit space of the induced action of $C^\infty_{+,x_0}(S^n)$ on $\mathcal{R}\mathrm{iem}(S^n)$. Clearly there is a canonical projection map $p_1 : \mathcal{C}_{x_0}(S^n) \longrightarrow \mathcal{C}(S^n)$ which is a homotopy equivalence since $p_1^{-1}(C) \cong \mathbf{R}$. Let $\mathcal{C}^+_{x_0}(S^n) = p_1^{-1}\left(\mathcal{C}^+(S^n)\right)$. We consider the following subgroup of the diffeomorphism group $\mathrm{Diff}_+(S^n)$:

$$\mathrm{Diff}_{x_0,+}(S^n) = \{\phi \in \mathrm{Diff}_+(S^n) \mid \phi(x_0) = x_0, \quad d\phi = Id : TM_{x_0} \to TM_{x_0}\}.$$

The group $\mathrm{Diff}_{x_0,+}(S^n)$ acts (on the right, by pulling back a metric) on the spaces $\mathcal{C}(S^n)$ and $\mathcal{C}_{x_0}(S^n)$. Then it is an easy observation that the group $\mathrm{Diff}_{x_0,+}(S^n)$ acts freely on the space $\mathcal{C}_{x_0}(S^n)$. Clearly the space $\mathcal{C}^+_{x_0}(S^n)$ of positive conformal classes is invariant under this action. We define the moduli space of positive conformal structures as the orbit space of the action of $\mathrm{Diff}_{x_0,+}(S^n)$ on $\mathcal{C}^+_{x_0}(S^n)$:

$$\mathcal{M}^+_{x_0,\mathrm{conf}}(S^n) = \mathcal{C}^+_{x_0}(S^n)/\mathrm{Diff}_{x_0,+}(S^n).$$

To make this construction usefull, we let $\widetilde{\mathrm{Diff}}_{x_0,+}(S^n) \subset \mathrm{Diff}_+(S^n)$ be yet another subgroup of diffeomorphisms ϕ with $\phi(x_0) = x_0$. The groups $\mathrm{Diff}_{x_0,+}(S^n)$, $\widetilde{\mathrm{Diff}}_{x_0,+}(S^n)$, and $\mathrm{Diff}_+(S^n)$ are clearly related to each other.† Indeed, one has the following fiber bundles:

$$\mathrm{Diff}_{x_0,+}(S^n) \longrightarrow \widetilde{\mathrm{Diff}}_{x_0,+}(S^n) \longrightarrow GL^+(n;\mathbf{R}),$$

$$\widetilde{\mathrm{Diff}}_{x_0,+}(S^n) \longrightarrow \mathrm{Diff}_+(S^n) \longrightarrow S^n$$

In particular, one concludes the isomorphisms:

$$\pi_0\mathrm{Diff}_{x_0,+}(S^n) \cong \pi_0\widetilde{\mathrm{Diff}}_{x_0,+}(S^n) \cong \pi_0\mathrm{Diff}_+(S^n) \cong \Theta^{n+1},$$

where $\Theta^{n+1} = \pi_0\mathrm{Diff}_+(S^n) \cong \pi_0\mathrm{Diff}_{x_0,+}(S^n)$ is the group of homotopy spheres. The space $\mathcal{C}_{x_0}(S^n)$ is contractible, thus the orbit space $\mathcal{C}_{x_0}(S^n)/\mathrm{Diff}_{x_0,+}(S^n)$ is homotopy equivalent to the classifying space $B\mathrm{Diff}_{x_0,+}(S^n)$. We obtain the following commutative diagram of fiber bundles:

$$\begin{tikzcd}
\mathcal{C}^+_{x_0}(S^n) \arrow[r, "\subset"] \arrow[d, "\mathrm{Diff}_{x_0,+}(S^n)"'] & \mathcal{C}_{x_0}(S^n) \arrow[d, "\mathrm{Diff}_{x_0,+}(S^n)"'] \\
\mathcal{M}^+_{x_0,\mathrm{conf}}(S^n) \arrow[r, "j"] & B\mathrm{Diff}_{x_0,+}(S^n)
\end{tikzcd}$$

† We are grateful to Thomas Schick for a clarifying discussion on that subject.

In particular, one has the exact sequence in homotopy groups:

$$\cdots \to \pi_1(\mathcal{C}^+_{x_0}(S^n)) \xrightarrow{p_*} \pi_1\mathcal{M}^+_{x_0,\mathrm{conf}}(S^n) \xrightarrow{\partial} \Theta^{n+1} \xrightarrow{i_*} \pi_0(\mathcal{C}^+_{x_0}(S^n)) \xrightarrow{p_*} \pi_0\mathcal{M}^+_{x_0,\mathrm{conf}}(S^n)$$

We think that the moduli space $\mathcal{M}^+_{x_0,\mathrm{conf}}(S^n)$ is an adequate model to study the positive scalar curvature metrics. It captures all homotopy properties of the standard moduli space $\mathcal{M}^+(S^n)$ of psc-metrics, and, on the other hand, is well-designed for conformal geometry. We conclude with the following challenging problem.

Problem 2. *Describe a rational homotopy type of the space* $\mathcal{M}^+_{x_0,\mathrm{conf}}(S^n)$.

References

[1] K. Akutagawa, B. Botvinnik, Relative Yamabe invariant, math.DG/0008138

[2] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[3] I. Chavel, Eigenvalues in Riemannian geometry, Pure and Applied Mathematics 115. Academic Press, 1984.

[4] P. Cherrier, Problémes de Neumann non linèaires sur les variètès riemanniennes, J. Funct. Anal. 57 (1984), 154–206.

[5] J. Escobar, The Yamabe problem on manifolds with boundary, J. Diff. Geom. 35 (1992), 21-84.

[6] P. Gajer, Concordances of metrics of positive scalar curvature, Pacific J. Math. 157 (1993), 257–268

[7] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order, Grundlehren der Mathematischen Wissenschaften, Vol. 224, Springer-Verlag, Berlin-New York, 1977.

[8] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures. Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993), 1–213, Progr. Math. 132, Birkhäuser Boston, Boston, MA, 1996.

[9] B. Hajduk, On the obstruction group to existence of Riemannian metrics of positive scalar curvature, Global differential geometry and global analysis (Berlin, 1990), 62–72, Lecture Notes in Math. 1481, Springer, Berlin, 1991.

[10] J. Jost, Harmonic mappings between Riemannian manifolds, Proc. Centre Math. Analysis 4, Australian National University Press, Canbera, 1984.

[11] J. Kazdan, F. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Diff. Geom. 10 (1975), 113–134.

[12] O. Kobayashi, Scalar curvature of a metric with unit volume, Math. Ann. 279 (1987), 253-265.

[13] J. Morava, H. Tamanoi, A vanishing theorem for the conformal anomaly in dimension > 2, Proc. Amer. Math. Soc. 100 (1987), 767–774.

[14] J. Rosenberg, S. Stolz, Metric of positive scalar curvature and connection with surgery, to appear.

[15] R. Schoen, S.-T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math. 92 (1988), 47–71.

[16] S. Stolz, Concordance classes of positive scalar curvature metrics, to appear.